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EXISTENCE OF GLOBAL SOLUTIONS OF THE MIXED PROBLEM FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH Q- LAPLACIAN OPERATORS

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Abstract. In the paper the mixed problem for the system of nonlinear q-Laplacian wave equations is studied. The theorems are proved on the existence of the global solutions of the considered problem.

Keywords: Wave equations, asymptotic behavior, global solutions, initial-boundary value problem, Laplacian operator, Holder's inequality, Young inequality.

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1 Introduction

The study of the existence and asymptotic behavior of global solutions of initial-boundary value problems for the wave equation with nonlinear operators such as

$$u_{tt} - \Delta_q u + (-\Delta)^{\alpha} u_t - h(u) = \varphi(t,x)$$

where $\Delta_q u = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^q \frac{\partial u}{\partial x_i}\right)$, $0 < \alpha < 1 |h(u)| \approx |u|^p$ is investigated in (Gao & Ma, 1999), (Hongjun & Hui, 2007).

For this problem, Gao & Ma (1999) (see also (Hongjun & Hui, 2007)) obtained the global existence of the solution when $q > p$ with small initial data when $q \leq p$. When $q = 2$, with the linear damping term $\alpha = 0$, Levine (1974) and (Levine, 1974) proved that solution blows up in the finite time with negative initial energy. When $q = 2$, and the damping term is given by $|u_t|^r u_t$, $r \geq 0$, many authors studied the existence and uniqueness of the global solution and the blowup of the solution, (see (Levine et al., 1997), (Georgiev & Todorova, 1994)).

In this paper we consider the nonlinear initial-boundary value problem

$$\begin{cases} u_{1tt} - \Delta_q u_1 + (-\Delta)^{\alpha} u_{1t} - f_1(u_1,u_2) = g_1(t,x), \\ u_{2tt} - \Delta_q u_2 + (-\Delta)^{\alpha} u_{2t} - f_2(u_1,u_2) = g_2(t,x), \end{cases} \tag{1}$$

$$u_j(t,x) = 0,\ t > 0,\ x \in \partial\Omega\,,\ j = 1,2, \tag{2}$$

$$u_j(0,x) = \varphi_j(x)\,,\ u_{jt}(0,x) = \psi_j(x)\,,\ \ x \in \Omega\,,\ \ j = 1,2. \tag{3}$$

Here Ω is a bounded domain in R^n, $n \geq 1$ with smooth boundary $\partial\Omega$, $t > 0$; $x \in \Omega$;$0 < \alpha \leq 1$,$f_1(u_1,u_2) = a_1 |u_1|^{\rho-1} |u_2|^{\rho+1} u_1$; $f_2(u_1,u_2) = a_2 |u_1|^{\rho+1} |u_2|^{\rho-1} u_2$;$g_1, g_1 : [0,T] \times \Omega \to$

$R;\Delta_q u = \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^q \frac{\partial u}{\partial x_i}\right); q \geq 2,\ a_1, a_2 \in R$ and $(-\Delta)^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha (u, \varphi_j)\varphi_j$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...; \varphi_1, \varphi_2, \varphi_3, ...$ are the sequence of eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, respectively.

The norm in $L_q(\Omega)$ is denoted by $\|\|_q$ and in $\overset{\circ}{W}{}_q^1(\Omega)$ we use the norm

$$\|u\|_{q,1}^q = \sum_{j=1}^{n} \| u_{x_j} \|_q^q .$$

We give some of the basic properties of the operators used here. The degenerate operator $\Delta_q u$ is bounded, monotone and hemicontinuous from $\overset{\circ}{W}{}_q^1(\Omega)$ to $W_{q'}^{-1}(\Omega)$, where $\frac{1}{q} + \frac{1}{q'} = 1$. The $(-\Delta)^\alpha$ powers for the Laplacian operator is defined by

$$(-\Delta)^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha (u, \varphi_j)\varphi_j .$$

We investigate the existence of the global solutions of problem (1) - (3).
In the case $q = 2,\ \alpha = 0$ this problem was studied in (Medeiros & Miranda, 1990), (Miranda & Medeiros, 1987), (Aliev & Yusifova, 2017), (Aliev & Rustamova, 2016), (Wang, 2009), (Ye, 2014).

2 Preliminaries and main results

Assume that

$$0 < \rho < \frac{nq}{2(n-q)} - 1,\ for\ n > q, \tag{4}$$

$$0 < \rho < +\infty\ \ for\ \ n \leq q. \tag{5}$$

To prove the existence of global solutions we use the Galerkin method.

Theorem 1. *Let conditions* (4)-(5) *hold, and* $\rho < \frac{q-1}{2}$. *Then for any* $u_{j0} \in \overset{\circ}{W}{}_q^1(\Omega)$, $u_{j1} \in L_2(\Omega)$ *and* $g_j(.) \in L_2([0,T] \times \Omega)$, $j = 1, 2$ *there exists a functions* $u_1, u_2 : [0,T] \times \Omega \to R$ *such that*

$$u_j \in L_\infty(0,T;\ \overset{\circ}{W}{}_p^1(\Omega)), \tag{6}$$

$$u_j' \in L_\infty(0,T;\ L_2(\Omega)) \bigcap L_2(0,T;\ D((-\Delta)^{\alpha/2})), \tag{7}$$

$$u_j(0) = u_{j0}, \quad u_j'(0) = u_{j1}\ ,\ \ j = 1, 2, \tag{8}$$

$$u_{j_{tt}} - \Delta_q u_j + (-\Delta)^\alpha u_{j_t} - f_j(u_1, u_2) = g_j(t,x)\ , j = 1, 2\,. \tag{9}$$

Next we consider an existence result when $\rho \geq \frac{q-1}{2}$. In this case, the global solution is obtained with small initial data.

Using the Holder and Young inequality we have

$$\int_\Omega |\varphi_{1m}\varphi_{2m}|^{\rho+1} dx \leq \frac{1}{2}\sum_{j=1}^{2}\int_\Omega |\varphi_{jm}|^{2(\rho+1)} dx \leq$$

$$\leq \frac{C_{2(\rho+1)}^{2(\rho+1)}}{2}\sum_{j=1}^{2}\left[\int_\Omega |\nabla\varphi_{jm}|^q dx\right]^{\frac{2(\rho+1)}{q}} \leq C_{2(\rho+1)}^{2(\rho+1)}\left[\sum_{j=1}^{2}\int_\Omega |\nabla\varphi_{jm}|^q dx\right]^{\frac{2(\rho+1)}{q}}, \tag{10}$$

where $C_{2(\rho+1)}$ is the Sobolev constant for the inequality

$$\|\varphi\|_{L_{2(\rho+1)}(\Omega)} \leq C_{2(\rho+1} \|\nabla\varphi\|_{L_{2(\rho+1)}(\Omega)}.$$

For each $m \in N$ we put

$$E_m = \sum_{j=1}^{2} \frac{1}{2a_j} \int_\Omega |\psi_{jm}|^2 \, dx + \frac{1}{qa_0} \sum_{j=1}^{2} \int_\Omega |\nabla\varphi_{jm}|^q \, dx + C_{2(\rho+1)}^{2(\rho+1)} \left[\sum_{j=1}^{2} \int_\Omega |\nabla\varphi_{jm}|^q \, dx \right]^{\frac{2(\rho+1)}{q}}, \quad (11)$$

where $a_0 = \min\{a_1, a_2\}$.

We also introduce the polynomial

$$Q(z) = \frac{1}{qa_0} z - C_{2(\rho+1)}^{2(\rho+1)} z^{\frac{2(\rho+1)}{q}}.$$

$Q(z)$ increases in $[0, z_0]$, where

$$z_0 = \left(\frac{1}{2a_0(\rho+1)C_{2(\rho+1)}^{2(\rho+1)}} \right)^{\frac{q}{2(\rho+1)-q}}.$$

Theorem 2. *Let conditions* (4),(5) *hold,* $\rho \geq \frac{q-1}{2}, u_{j0} \in \overset{\circ}{W}{}_q^1(\Omega)$, $u_{j1} \in L_2(\Omega)$ *and* $g_j(.) \in L_2([0,T] \times \Omega)$, $j = 1, 2$. *Suppose inaddition that initial data satisfy the following conditions*

$$\sum_{j=1}^{2} \|\nabla\varphi_j\|_{L_q(\Omega)}^q < z_0, \quad (12)$$

$$E_0 + \frac{1}{4\lambda_1^\alpha} \sum_{j=1}^{2} \int_0^T \int_\Omega |g_j(t,x)|^2 \, dx < Q(z_0), \quad (13)$$

where $E_0 = \lim\limits_{m\to+\infty} E_m$. *Then there exists the functions* $u_1, u_2 : [0,T] \times \Omega \to R$ *satisfying* (6)-(9).

Proof of Theorem 1. Let r be an integer for which $H_0^r(\Omega) \subset \overset{\circ}{W}{}_0^1(\Omega)$ is continuous. Then the eigenfunctions of $-\Delta^r w_k = \lambda_k w_k$ in $H_0^r(\Omega)$ yields a Galerkin basis for both $H_0^r(\Omega)$ and $L_2(\Omega)$. For each $m \in N$, let us put $V_m = Span\{w_1, w_2, ..., w_n\}$.

We search for the function

$$u_{jm}(t) = \sum_{k=1}^{m} h_{j_{km}}(t) w_k \ , \ \ j = 1, 2,$$

that for any $v \in V_m$, $u_{jm}(t)$ satisfies the approximate equation

$$\int_\Omega \left\{ \, u''_{j_m} - \Delta_q u_{j_m} + (-\Delta)^\alpha u'_{j_m} - f_j\left(u_{1_m}, u_{2_m}\right) - g_j(t,x) \right\} \, v dx = 0, \quad (14)$$

with the initial conditions

$$u_{j_m}(0) = \varphi_{j_k}, \qquad u'_{j_m}(0) = \psi_{j_m}, \quad (15)$$

where $j = 1, 2, \ m = 1, 2, ..., \varphi_{j_m}$ and ψ_{j_m} are chosen in V_m such that

$$\varphi_{j_m} \to \varphi_j \ in \ \overset{\circ}{W}{}_q^1(\Omega) \ and \ \psi_{j_m} \to \psi_j \ in \ L_2(\Omega), \ j = 1, 2. \quad (16)$$

Taking $v = w_k, \ \ k = 1, 2, ..., m,$ we see that

$$\int_\Omega \left\{ \ u''_{j_m} - \Delta_q u_{j_m} + (-\Delta)^\alpha u'_{j_m} - f_j(u_{1_m}, u_{2_m}) - g_j(t,x) \right\} \ w_k \, dx = 0 \tag{17}$$

has a local solution $(u_{1m}(t), u_{2m}(t))$ in the interval $[0, T_m)$. In the next step we obtain a'priori estimates for the solution $(u_{1m}(t), u_{2m}(t))$ that can be extended to the whole interval $[0, T]$.

A priori estimates: Multiplying both sides of (17) by $\frac{1}{a_j} u'_{j_{km}}(t)$ and summing the obtained equalities in $k = 1, 2, ..., m$ and then integrating we have

$$\begin{gathered}\frac{1}{2a_j} \int_\Omega \left|u'_{j_m}\right|^2 dx + \frac{1}{qa_j} \ \int_\Omega \left|\nabla u_{j_m}\right|^q dx + \\ + \frac{1}{a_j} \int_0^t \int_\Omega \left|\nabla^\alpha u'_{j_m}\right|^2 \, dxdt - \int_0^t \int_\Omega f_j(u_{1_m}, u_{2_m}) \ u'_{j_m} dxdt = \\ = \frac{1}{2a_j} \int_\Omega \left|\psi_{j_m}\right|^2 dx + \frac{1}{qa_j} \int_\Omega \left|\nabla \varphi_{j_m}\right|^q dx + \frac{1}{a_j} \int_0^t \int_\Omega g_j(t,x) \ u'_{j_m}(t,x) dxdt.\end{gathered} \tag{18}$$

On the other hand

$$\begin{gathered}\sum_{j=1}^2 \int_0^t \int_\Omega f_j(u_{1_m}, u_{2_m}) \ u'_{j_m} dxdt = \\ = \int_0^t \int_\Omega \left[\left|u_{1m}\right|^{\rho-1} \left|u_{2m}\right|^{\rho+1} u_{1m} u'_{1m} + \left|u_{1m}\right|^{\rho+1} \left|u_{2m}\right|^{\rho-1} u_{2m} u'_{2m} \right] \, dx \ = \\ = \int_\Omega \left|u_{1m}\right|^{\rho+1} \left|u_{2m}\right|^{\rho+1} dx \ \ - \ \int_\Omega \left|\varphi_{1m}\right|^{\rho+1} \left|\varphi_{2m}\right|^{\rho+1} dx.\end{gathered} \tag{19}$$

From (6) and (7) it follows that

$$\begin{gathered}\sum_{j=1}^2 \left[\frac{1}{2a_j} \int_\Omega \left|u'_{j_m}\right|^2 dx + \frac{1}{qa_j} \ \int_\Omega \left|\nabla u_{j_m}\right|^q dx + \frac{1}{a_j} \int_0^t \int_\Omega \left|\nabla^\alpha u'_{j_m}\right|^2 \, dxdt \right] \leq \\ \leq \sum_{j=1}^2 \frac{1}{2a_j} \int_\Omega \left|\psi_{j_m}\right|^2 dx + \frac{1}{qa_j} \int_\Omega \left|\nabla \varphi_{j_m}\right|^q dx + \ \int_\Omega \left|\varphi_{1m}\right|^{\rho+1} \left|\varphi_{2m}\right|^{\rho+1} dx \ + \\ + \int_\Omega \left|u_{1m}\right|^{\rho+1} \left|u_{2m}\right|^{\rho+1} dx + \sum_{j=1}^2 \frac{1}{a_j} \int_0^t \int_\Omega g_j(t,x) \ u'_{j_m}(t,x) dxdt\end{gathered} \tag{20}$$

Applying Holder's inequality we get

$$\begin{gathered}\int_\Omega \left|u_{1m}\right|^{\rho+1} \left|u_{2m}\right|^{\rho+1} dx \ \ \leq \left(\int_\Omega \left|u_{1m}\right|^{2(\rho+1)} dx \ \right)^{1/2} \cdot \ \left(\int_\Omega \left|u_{2m}\right|^{2(\rho+1)} dx \ \right)^{1/2} \\ \leq \ \int_\Omega \left|u_{1m}\right|^{2(\rho+1)} dx \ + \ \int_\Omega \left|u_{2m}\right|^{2(\rho+1)} dx\end{gathered}$$

Using embedding theorems (Lions & Majens, 1969), we obtain

$$\int_\Omega \left|u_{1m}\right|^{\rho+1} \left|u_{2m}\right|^{\rho+1} dx \ \ \leq A_0 \sum_{>=1}^2 \left(\int_\Omega \left|\nabla u_{j_m}\right|^q dx \right)^{2(\rho+1)}.$$

Further, using the Young inequality, we have the following inequality

$$\int_{\Omega} |u_{1m}|^{\rho+1} |u_{2m}|^{\rho+1} \, dx \;\; \leq \varepsilon \sum_{>=1}^{2} \int_{\Omega} |\nabla u_{j_m}|^q \, dx + \frac{A_0 q}{2\varepsilon(\rho+1)}, \tag{21}$$

where $0 < \varepsilon < \min\left\{ \frac{1}{qa_1}, \frac{1}{qa_2} \right\}$.
From (16), (20) and (21) we obtain an a'priori estimate

$$\sum_{j=1}^{2} \frac{1}{a_j} \left[\frac{1}{2} \int_{\Omega} \left| u'_{j_m} \right|^2 dx + \frac{1}{q} \int_{\Omega} |\nabla u_{j_m}|^q \, dx + \int_0^t \int_{\Omega} \left| \nabla^{\alpha} u'_{j_m} \right|^2 dx dt \right] \leq \; A, 0 \leq t \leq T. \tag{22}$$

If follows that the solutions of the approximated problem (14), (15) can be extended to [0,T].

Considering (22) we have

$$\{u_{jm}\} \; is \; bounded \; in \; L_{\infty}(0,T; \overset{0}{W}{}^1_q(\Omega)), \; j = 1,2; \tag{23}$$

$$\{u'_{jm}\} \; is \; bounded \; in \; L_{\infty}(0,T; L_2(\Omega)), \; j = 1,2; \tag{24}$$

$$\{\Delta^{\alpha} u'_{jm}\} \; is \; bounded \; in \; L_2(0,T; L_2(\Omega)), \; j = 1,2; \tag{25}$$

$$\{-\Delta_q u_{jm}\} \; is \; bounded \; in \;\; L_{\infty}(0,T; W^{-1}_{q'}(\Omega)), \; j = 1,2. \tag{26}$$

(see (Lions, 1969), (Lions & Majens, 1969)).

Since $H^2(\Omega) \subset \overset{0}{W}{}^1_q(\Omega))$, from (26) follows that

$$\{u''_{jm}\} \; is \; bounded \; in \; L_2(0,T; H^{-r}(\Omega)). \tag{27}$$

Taking into account (23)-(27), from the sequence $\{(u_{1m}, u_{2m})\}$ we can select a subsequence $\{(u_{1\mu}, u_{2\mu})\}$ such that

$$u_{j\mu} \to u_j \; weakly \; in \; L_{\infty}(0,T; \overset{0}{W}{}^1_q(\Omega)), \tag{28}$$

$$u'_{j\mu} \to u'_j \; weakly \; in \; L_{\infty}(0,T; L_2(\Omega)), \tag{29}$$

$$\Delta^{\frac{\alpha}{2}} u'_{j\mu} \to \Delta^{\frac{\alpha}{2}} u'_j \; weakly \; in \; L_2(0,T; L_2(\Omega)), \tag{30}$$

$$\Delta_q u_{j\mu} \to \chi_j \; weakly \; in \; L_{\infty}(0,T; W^{-1}_{q'}(\Omega)). \tag{31}$$

In view of the continuity of the embedding

$$W^1_2 \left(0, T; \overset{0}{W}{}^2_1(\Omega), L_2(\Omega) \right) \subset C\left([0,T]; L_2(\Omega)\right)$$

(see (Lions & Majens, 1969)), it follows from (28), (29) that

$$u_{jm} \to u_j \; strongly \; in \; C([0,T]; L_2(\Omega)), j = 1,2. \tag{32}$$

From (28), (29) follows that

$$u'_{jm} \to u'_j \; in \; C_w([0,T]; L_2(\Omega)), \tag{33}$$

i.e. $\left\langle u'_{jm},\nu\right\rangle \to \langle u_j,\nu\rangle$ for any $\nu \in L_2(\Omega)$(see (Lions & Majens, 1969)).

On the other hand due to compactness of the embedding $D\left((-\Delta)^{\frac{\alpha}{2}}\right) \subset L_2(\Omega)$, from (29) and (30) we obtain that

$$u'_{jm} \to u'_j \textit{ strongly in } L_2(0,T;\, L_2(\Omega)),\ j = 1,2. \tag{34}$$

(see[14]).

Taking into account conditions (4),(5) from (28) we have

$$\int_0^T \int_\Omega \left| |u_{1m}|^{\rho-1} |u_{2m}|^{\rho+1} u_{1m} \right|^{\frac{2(\rho+1)}{2\rho+1}} dxdt \ \le C, \tag{35}$$

$$\int_0^T \int_\Omega \left| |u_{1m}|^{\rho+1} |u_{2m}|^{\rho-1} u_{2m} \right|^{\frac{2(\rho+1)}{2\rho+1}} dxdt \ \le C. \tag{36}$$

On the other hand, from (32) we have

$$|u_{1m}|^{\rho-1} |u_2|^{\rho+1} u_{1m} \to |u_1|^{\rho-1} |u_2|^{\rho+1} u_1 \ , a.e.\ in\ [0,T] \times \Omega, \tag{37}$$

$$|u_{1m}|^{\rho\ +1} |u_{2,}|^{\rho\ -1} u_{2m} \to |u_1|^{\rho+1} |u_2|^{\rho-1} u_2,\ a.e.\ in\ [0,T] \times \Omega. \tag{38}$$

Then using Lemma 3.1 from Lions (1969) in (29)-(38) we get

$$f_j(u_{1m},u_{2m}) \to f_j(u_1,u_2) \textit{ weakly in } L^{\frac{2(\rho+1)}{2\rho+1}}\left(0,T; L_{\frac{2(\rho+1)}{2\rho+1}}(\Omega)\right),\ j = 1,2. \tag{39}$$

With the convergence of (28)-(38) we can pass to limit in the approximate equations and get

$$\frac{d}{dt}\left(u_j^{'}(t),\nu_j\right) + \langle \chi_j(t),\nu_j\rangle + \left((-\Delta)^{\alpha} u'_j(t),\nu_j\right) \ - (f_j(u_1,u_2),\nu_j) = (g_j(t,\cdot),\nu_j)\ ,\ j = 1,2 \tag{40}$$

for all $\nu_1,\nu_2 \in W_q^1(\Omega)$, in the sense of distributions.

In (40) we set $\nu_j = u_{jm}(t), j = 1,2$ and integrate both sides of the obtained inequality. Then we have

$$\int_0^T (\Delta_q u_{jm}(t),u_{jm}(t))\,dt = \left(u'_{jm}(T),u_{jm}(T)\right) - \left(u'_{jm}(0),u_{jm}(0)\right) +$$

$$+\int_0^T \left((-\Delta)^{\alpha/2} u'_{jm}(t),(-\Delta)^{\alpha/2} u_{jm}(t)\right) dt - \int_0^T (f_j(u_{1m},u_{2m}),u_{jm}(t))\,dt = \tag{41}$$

$$= \int_0^T (g_j(t,\cdot),u_{jm}(t))\,dt.$$

From (32), (33) we obtain

$$\left(u'_{jm}(\tau),u_{jm}(\tau)\right) \to \left(u'_j(T),u_j(T)\right)\ , \tag{42}$$

$$\left(u'_{jm}(0),u_{jm}(0)\right) \to \left(u'_j(0),u_j(0)\right)\ . \tag{43}$$

Then from (40)-(43) we get

$$\lim_{\mu\to\infty} \int_0^T (\Delta_q u_{j\mu}(t),u_{j\mu}(t))\,dt \le \int_0^T (\chi_j(t),u_j(t))\,dt.$$

From monotonicity of $-\Delta_q$ follows that

$$\chi_j(t) = \Delta_q u_j(t) \tag{44}$$

(see (Lions, 1969)).

Thus, from (40) and (44) it follows that $(u_1(t)\,,\, u_2(t))$ is a solution of problem (1)-(3).

Proof of Theorem 2. From (20) we obtain

$$\sum_{j=1}^{2} \frac{1}{2a_0} \int_\Omega \left| u'_{jm}(t,x)^2 \right| dx + Q\left(\sum_{j=1}^{2} \|\nabla u_{jm}(t,x)\|^q_{L_q(\Omega)} \right) +$$

$$+ \sum_{j=1}^{2} \int_0^T \int_\Omega \left| \Delta^{\frac{\alpha}{2}} u'_j(t,x) \right|^2 dxdt \le \tag{45}$$

$$\le E_n + \sum_{j=1}^{2} \int_0^T \int_\Omega f_j(t,x) u_{jm}(t,x) dxdt.$$

Since $\int_\Omega \left|\Delta^{\frac{\alpha}{2}} u'_j(t,x)\right|^2 dx \ge \lambda_1^\alpha \int_\Omega \left|u'_j(t,x)\right|^2 dx$, then from (45) we have

$$\sum_{j=1}^{2} \frac{1}{2a_0} \int_\Omega \left| u'_{jm}(t,x) \right|^2 dx + Q\left(\sum_{j=1}^{2} \|\nabla u_{jm}(t,x)\|^q_{L_q(\Omega)} \right) \le$$

$$\le E_n + \frac{1}{4\lambda_1^\alpha} \sum_{j=1}^{2} \int_0^T \int_\Omega |f_j(t,x)|^2 dxdt. \tag{46}$$

Lemma 1. *There exists a number N, such that for any $m > N$*

$$\sum_{j=1}^{2} \|\nabla u_{jm}(t,x)\|^q_q < z_0 \tag{47}$$

is valid .

On the other hand if $0 < z < z_0$ then

$$0 \le Q(z) \le Q(z_0). \tag{48}$$

Then, by Lemma 1

$$Q\left(\sum_{j=1}^{2} \|\nabla u_{jm}(t,x)\|^q_{L_q(\Omega)} \right) \ge 0. \tag{49}$$

Then from (46) and (49) follows that

$$\sum_{j=1}^{2} \frac{1}{2a_0} \int_\Omega \left| u'_{jm}(t,x) \right|^2 dx \le C, \quad t \in [0, t_m]. \tag{50}$$

By (47) and (50), a priori estimate (22) holds.

Proof of Lemma 1. Suppose (46) is false. Then for each $m > N$, there exists $t \in [0, t_m]$ such that

$$\sum_{j=1}^{2} \|\nabla u_{jm}(t)\|^q_{L_q(\Omega)} \ge z_0, \quad \forall m > N_0. \tag{51}$$

By virtue of (12) and (16), there exists N_0 for which

$$\sum_{j=1}^{2} \|u_{jm}(0)\|_{L_q(\Omega)}^q < z_0 \quad \forall m > N_0$$

Then by continuity of $\|u_{jm}(t)\|_q^q$ there exists $t_m^\alpha \in [0, t_m]$ such that

$$\sum_{j=1}^{2} \|\nabla u_{jm}(t_m^\alpha)\|_{L_q(\Omega)}^q = z_0,$$

where

$$Q\left(\sum_{j=1}^{2} \|\nabla u_{jm}(t)\|_{L_q(\Omega)}^q\right) \geq 0, \quad t \in [0, t_m^\alpha]. \tag{52}$$

From (48) and (52) there exists $N > N_0$ and $\beta \in (0, z_0)$ such that

$$0 \leq \frac{1}{2}\left\|u_m'(t)\right\|^2 + Q\left(\sum_{j=1}^{2} \|u_{jm}(t)\|_q^q\right) \leq Q(\beta), \forall t \in [0, t_m^\alpha],\ \forall m > N.$$

Considering the monotonicity of $Q(z)$ in $[0, z_0]$ we get,
$0 \leq \sum_{j=1}^{2} \|u_{jm}(t)\|_q^q \leq \beta < z_0$, $\forall t \in [0, t_m^\alpha]$ and in particular $\sum_{j=1}^{2} \|u_{jm}(t_m^\alpha)\|_q^q < z_0$ that contradicts to(51).

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